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On Dirac's methods for constrained systems and gauge-fixing conditions with explicit time dependence

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It is shown that Dirac's methods for constrained hamiltonian systems require careful application if gauge-fixing conditions with explicit time dependence are used. It is non-trivial, in general, to find both an appropriate set of variables for the physical phase space and a hamiltonian furnishing the correct time evolution of these quantities. This problem is solved for a certain class of gauge-fixing conditions. As well as offering the possibility of new applications the solution provides a more systematic understanding of some well-known cases, among which are the light-cone and temporal gauges for relativistic particles and strings.

Due to the overwhelming importance of gauge symmetries in theoretical high-energy physics, Dirac's elegant methods for constrained hamiltonian systems [1] have been widely studied and applied [2]. In addition string theory has provided us with a very well-known example of a gauge condition with explicit time dependence, namely the light-cone gauge [3,4]. Despite this, however, there appears to have been no thorough study of more general time-dependent gauge-fixing conditions within Dirac's formalism. In this paper we point out some difficulties which can arise in such circumstances and we show how these problems can be overcome for a wide class of examples.

We shall assume a familarity with Dirac's methods, as discussed in ref. [1] or ref. [2]. Consider a dynamical system with a phase space Γ described by $2d$ canonical variables $z_M=(q_\mu, p_\mu)$ and classical Poisson brackets

$$[f,g]=\frac{\partial f}{\partial q_\mu}\frac{\partial g}{\partial p_\mu}-\frac{\partial g}{\partial q_\mu}\frac{\partial f}{\partial p_\mu}. \tag{1}$$

Let H be the hamiltonian so that the evolution in time τ of any function $f(z_M, \tau)$ is given by

$$\dot f\equiv\frac{\mathrm{d}f}{\mathrm{d}\tau}=\frac{\partial f}{\partial\tau}+[f,H]. \tag{2}$$

We shall consider exclusively systems possessing constraints $\phi_i(z_M)$, $i=1, ..., n$, which are first-class, obeying a closed algebra

$$[\phi_i,\phi_j]\simeq 0, \tag{3}$$

and which are preserved in time:

$$\frac{\mathrm{d}\phi_i}{\mathrm{d}\tau}=[\phi_i, H]\simeq 0. \tag{4}$$

The symbol $\simeq$ denotes equality up to terms which vanish when $\phi_i=0$.

The ϕ_i generate gauge symmetries acting via the Poisson bracket and these symmetries manifest themselves as ambiguities in the time evolution of any quantity. According to Dirac's prescription, H contains a sum of all primary first-class constraints, each one multiplied by some arbitrary function of τ and z_M. Different choices of these functions yield different trajectories on Γ with the same initial conditions, the trajectories being related by gauge transformations. A function f is defined to be gauge-invariant if

$$[\phi_i, f]\simeq 0, \tag{5}$$

since then the time evolution of f will be independent of the ambiguities inherent in the definition of H [5]. It is clear that there are $2d-n$ independent gauge-invariant quantities, n of which are the constraints themselves.

Let us recall how one can gauge fix such a system

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and restrict its dynamics to a physical phase space $\Gamma^* \subset \Gamma$. One must select n gauge-fixing conditions χ_i with the property that $[\phi_i, \chi_j]$ is a non-singular matrix. It is convenient to set $\psi_I = (\chi_i, \phi_i)$, then Γ^* is the space of dimension $2d-2n$ defined by $\psi_I = 0$. One must choose the functions in H to ensure that the gauge conditions are preserved in time,

$$\frac{d\chi_i}{d\tau} = \frac{\partial \chi_i}{\partial \tau} + [\chi_i, H] \approx 0 , \tag{6}$$

where the symbol $\approx$ denotes equality up to terms which vanish on Γ^*. This equation together with (4) may be written $d\psi_I/d\tau \approx 0$ which implies that any trajectory starting in Γ^* remains in Γ^* for all time.

To introduce a bracket on Γ^* we define the Dirac bracket of functions on Γ by

$$[f, g]^* = [f, g] - [f, \psi_I](\Omega^{-1})_{IJ}[\psi_J, g] ,$$
$$\Omega_{IJ} = [\psi_I, \psi_J] . \tag{7}$$

It is bilinear and can be shown to satisfy the Jacobi identity. Furthermore, by construction any Dirac bracket involving ψ_I vanishes identically. Consequently we can consistently set $\psi_I = 0$ and obtain a well-defined induced bracket on Γ^*.

To describe the dynamics on Γ^* intrinsically, and so complete the gauge fixing process, we must write a time evolution equation for the system similar to (2) but using the Dirac bracket. If the χ_i have no explicit time dependence this is easily accomplished: for any function $f(z_M, \tau)$ we have

$$\frac{df}{d\tau} \approx \frac{\partial f}{\partial \tau} + [f, H]^* , \tag{8}$$

which restricts to an exact equation on Γ^*. To establish this we need only compare the right-hand sides of (2) and (8). Consider the block forms

$$\psi = \begin{pmatrix} \chi \\ \phi \end{pmatrix} ,$$

$$\Omega \approx \begin{pmatrix} \alpha & \omega \\ -\omega^{\mathrm{T}} & 0 \end{pmatrix} , \quad \Omega^{-1} \approx \begin{pmatrix} 0 & -(\omega^{-1})^{\mathrm{T}} \\ \omega^{-1} & \omega^{-1}\alpha(\omega^{-1})^{\mathrm{T}} \end{pmatrix} , \tag{9}$$

where we have used (3) and where the second equation defines the matrices α_{ij} and ω_{ij}. From these and from (4) we have

$$[f, H]^* - [f, H] \simeq -[f, \phi_i](\omega^{-1})_{ij}[\chi_j, H] . \tag{10}$$

So far we have assumed nothing about the nature of the gauge-fixing conditions. Now, if the χ_j have no explicit time dependence, (6) implies $[\chi_j, H] \approx 0$ on the right-hand side above, and so (8) does indeed hold for all f.

If the gauge conditions are explicitly time-dependent the situation is rather more subtle because there is now time dependence in the embedding $\Gamma^* \subset \Gamma$ itself. The first step in looking for an equation analogous to (8) is a more detailed consideration of this embedding. It is useful to introduce the notion of a set of physical [#1] variables $z^*_A = (q^*_a, p^*_a)$ $a = 1, ..., d-n$ defined to be smooth functions on Γ which when $\psi_I = 0$ provide a set of coordinates for Γ^*. In other words, locally in any region of Γ intersecting Γ^* we have a smooth (although not canonical) change of coordinates

$$\{z^*_A, \psi_I\} \leftrightarrow \{z_M\} . \tag{11}$$

With an abuse of notation we will write $z^*_A(z_M, \tau)$ to denote a particular choice of physical variables. Notice that these functions may or may not depend explicitly on time, whereas the full transformation (11) always contains τ explicitly when ψ_I does.

To describe dynamics on Γ^* we shall seek an equation on Γ

$$\frac{df}{d\tau} \approx \frac{\partial f}{\partial \tau} + [f, H^*]^* , \tag{12}$$

holding for any function of some chosen set of physical variables $f(z^*_A, \tau)$. The key point is that the hamiltonian H^* appearing in this equation will differ from the original hamiltonian H, in general. Moreover, different choices of the physical variables will require different choices of H^*. The most straightforward way to understand this is to note that the partial τ-derivatives in (2) and (12) are defined with z_M and z^*_A held fixed respectively. But the transformation (11) between these quantities involves τ explicitly so that one cannot expect to deduce (12) with $H^* = H$ from (2), in general. Similar considerations apply on changing from one set of physical variables to another. This is in marked contrast to our previous discussion of time-independent gauge fixing where the choice of physical variables was entirely implicit and the original hamiltonian appeared in (8) for any chosen set. We can now see that these features arise because the transformation (12) is τ-independent in this case (we tacitly allowed only those $z^*_A(z_M)$ with no τ dependence).

After these preparatory remarks the most general question we can ask is the following. Given a dynamical system and some set of gauge-fixing conditions, for what choices of physical variables can we find a hamiltonian H^* ensuring (12)? In the remainder of this paper we present some answers to this question in various cases. The first result we establish is valid for any type of gauge-fixing condition.

(A) We can always choose $z^*_A(z_M)$ to be τ-independent, *gauge-invariant* functions, and then $H^* = H$. The defining equation (5) involves no τ dependence, so we may assume its solutions are τ-independent. We have already remarked that there are $2d-n$ independent gauge-invariant functions on Γ, n of which are the ϕ_i, therefore there are $2d-2n$ of these quantities which remain independent when we restrict to Γ^*, the required number for a set of physical variables. To show $H^* = H$ we start from (2) and proceed as before to reach (10). Since χ_j has explicit time dependence $[\chi_j, H] \not\approx 0$. However, for any $f(z^*_A, \tau)$ with z^*_A gauge-invariant $[f, \phi_i] \simeq 0$ in (10) and so we deduce (12) with $H^* = H$.

To illustrate this result, as well as some of the remarks preceeding it, consider a relativistic particle of mass $m > 0$ described by the usual reparametrization-invariant action [#2]

$$S = -m \int d\tau (-\dot{q}_\mu \dot{q}^\mu)^{1/2} . \tag{13}$$

There is a single primary first-class constraint $\phi = \frac{1}{2}(p^2 + m^2)$, the hamiltonian is $H = \frac{1}{2}u(p^2 + m^2)$ for some function $u(\tau, q^\mu, p_\mu)$ and the resulting equations of motion are

$$\dot{q}^\mu \simeq u p^\mu , \quad \dot{p}_\mu \simeq 0 . \tag{14}$$

An appropriate set of gauge-invariant functions on Γ is given by

$$q^*_a = q_a - \frac{q_0}{p_0} p_a , \quad p^*_a = p_a , \quad a = 1, ..., d-1 , \tag{15}$$

and these clearly obey

$$\dot{q}^*_a \simeq 0 , \quad \dot{p}^*_a \simeq 0 . \tag{16}$$

To fix the reparametrization symmetry we choose the temporal gauge

$$\chi = q^0 - \tau , \tag{17}$$

and to preserve this we must take $u = 1/p^0$. On restricting to Γ^* we have $\phi = 0$ implying $H = 0$ which does indeed give the correct equations of motion for (q^*_a, p^*_a). Notice, however, that $H = 0$ certainly does not give the correct equations for the alternative set of physical variables (q_a, p_a).

Although result (A) is sufficient to deal with any example of time-dependent gauge fixing in principle, from a practical point of view it is unsatisfactory in a number of respects. Given some general dynamical system it would be a complicated task to find all its gauge-invariant quantities explicitly and even having done this it would then be highly inconvenient to be forced to work with these alone. This point is particularly relevant to field theory applications for which one expects any formulation using solely gauge-invariant quantities to be non-local. Motivated by these objections, it is natural to consider under what circumstances we can find more convenient sets of physical variables together with their corresponding hamiltonians (in the case of the particle, for example, one would prefer to work directly with (q_a, p_a)). The following result describes how this can be achieved for a reasonably general class of gauge conditions.

(B) Suppose we can divide the canonical variables (q_μ, p_μ) on Γ into disjoint sets (Q_i, P_i) $i = 1, ..., n$ and (q_a, p_a) $a = 1, ..., d-n$ so that the gauge-fixing conditions can be written

$$\chi_i = Q_i - \zeta_i(q_a, \tau) \tag{18a}$$

for certain functions ζ_i. Then a set of physical variables together with their associated hamiltonian is given by

$$q^*_a = q_a , \quad p^*_a = p_a + \frac{\partial \zeta_i}{\partial q_a} P_i , \quad H^* = H - \frac{\partial \zeta_i}{\partial \tau} P_i . \tag{18b}$$

[#1] This should not be confused with the notion of a gauge-invariant quantity previously defined by (5). The relationship between these concepts will be clarified in result (A).

[#2] $\mu = 0, ..., d-1$ and the metric convention is "mostly plus".

In the above one should think of (Q_i, P_i) as the variables being eliminated by gauge fixing. The idea behind the result is that the change of variables from (q_a, p_a) to (q_a^*, p_a^*) will, for a wide class of gauge choices, be significantly simpler than the change to gauge-invariant variables necessary to apply (A). It is also worth stating explicitly the following special case.

(C) With conditions as in (B) but with

$$\chi_i = Q_i - \zeta_i(\tau) \tag{19a}$$

(no q_a dependence in ζ_i) a set of physical variables and their associated hamiltonian is given by

$$q_a\,,\quad p_a\,,\quad H^* = H - \dot{\zeta}_i(\tau) P_i\,. \tag{19b}$$

To prove result (B) we begin by noting that the assumed forms for the gauge conditions imply that they commute, so there exists a canonical transformation to new variables, denoted by tildes, with $\tilde{Q}_i = \chi_i$. The desired generating function is

$$\mathscr{F}(q_\mu, \tilde{p}_\mu, \tau) = q_a \tilde{p}_a + [Q_i - \zeta_i(q_a, \tau)]\tilde{P}_i\,, \tag{20a}$$

from which we can deduce the complete change of variables

$$\tilde{Q}_i = \frac{\partial \mathscr{F}}{\partial \tilde{P}_i} = Q_i - \zeta_i\,,\quad \tilde{q}_a = \frac{\partial \mathscr{F}}{\partial \tilde{p}_a} = q_a\,,$$

$$P_i = \frac{\partial \mathscr{F}}{\partial Q_i} = \tilde{P}_i\,,\quad p_a = \frac{\partial \mathscr{F}}{\partial q_a} = \tilde{p}_a - \frac{\partial \zeta_i}{\partial q_a}\tilde{P}_i\,, \tag{20b}$$

together with the new hamiltonian

$$\tilde{H} = H + \frac{\partial \mathscr{F}}{\partial \tau} = H - \frac{\partial \zeta_i}{\partial \tau}\tilde{P}_i\,. \tag{20c}$$

When written as functions of the new variables the gauge-fixing conditions χ_i have no explicit τ dependence, while the constraints ϕ_i become τ-dependent, in general. We can now repeat the arguments used to establish result (A) but with the new variables replacing the old, and the roles of ϕ_i and χ_i interchanged. More precisely, we start from (2) written for the tilded variables, then using the block forms (10) with $\alpha_{ij}=0$ together with $[\chi_i, \tilde{H}] \approx 0$ we deduce

$$[f, \tilde{H}]^* - [f, \tilde{H}] \simeq [f, \chi_i](\omega^{-1})_{ji}[\phi_j, \tilde{H}] \tag{21}$$

for any f. Since $\chi_i = \tilde{Q}_i$, the right-hand side will vanish if we restrict $f(\tilde{q}_a, \tilde{p}_a, \tau)$. Taking $q_a^* = \tilde{q}_a$, $p_a^* = \tilde{p}_a$ and $H^* = \tilde{H}$ we deduce (12), completing the proof of (B).

For a simple application of (C) we return to the relativistic particle with the temporal gauge choice (17). Choosing (q_a, p_a) as physical variables the constraint $\phi = 0$ can be "solved"

$$p^0 = (p_a^2 + m^2)^{1/2} \tag{22}$$

and the equations of motion are, after gauge fixing,

$$\dot{q}_a = (p_a^2 + m^2)^{-1/2} p_a\,,\quad \dot{p}_a = 0\,. \tag{23}$$

One can readily verify that these follow from the hamiltonian given by (C):

$$H^* = -p_0 = (p_a^2 + m^2)^{1/2}\,. \tag{24}$$

This expression for H^* is not quite as obvious as it may seem. Given the gauge choice $x^0 = \tau$ one might guess that the desired hamiltonian should be p_0, since this is the momentum conjugate to x^0 and the hamiltonian should generate translations in τ. In fact the treatment above shows that this naive guess is wrong by a sign. In this particular case it would not be difficult to find the correct answer by inspection. Nevertheless these remarks emphasize the desirability of a more systematic approach as developed here.

A more substantial example, requiring the use of (B), is provided by the light-cone gauge in string theory [3,4]. Consider a closed string whose motion $q^\mu(\tau, \sigma)$ $0 \leqslant \sigma \leqslant 2\pi$ is described by the Nambu–Goto action [#3]

$$S = -\frac{1}{2\pi}\int d\tau \int_0^{2\pi} d\sigma \left[\left(\frac{\partial q^\mu}{\partial \tau}\frac{\partial q_\mu}{\partial \sigma}\right)^2 - \left(\frac{\partial q^\mu}{\partial \tau}\right)^2 \left(\frac{\partial q^\mu}{\partial \sigma}\right)^2\right]^{1/2}\,. \tag{25}$$

Introducing conjugate momenta $p_\mu(\tau, \sigma)$ one finds constraints

$$\phi_1 = p_\mu^2 + \frac{1}{(2\pi)^2}\left(\frac{\partial q^\mu}{\partial \sigma}\right)^2\,,\quad \phi_2 = p_\mu \frac{\partial q^\mu}{\partial \sigma}\,. \tag{26}$$

As with the particle, the hamiltonian H is a linear combination of these constraints and so will vanish on gauge fixing. The position zero-mode of the string and its total momentum are conjugate variables which we shall denote by

$$\mathscr{Q}^\mu(\tau) = \frac{1}{2\pi}\int_0^{2\pi} d\sigma\, q^\mu(\tau, \sigma)\,,$$

$$\mathscr{P}_\mu(\tau) = \int_0^{2\pi} d\sigma\, p_\mu(\tau, \sigma)\,, \tag{27}$$

respectively. The light-cone gauge is then defined by

$$\chi_1 = q^+ - \mathscr{P}^+\tau\,,\quad \chi_2 = p^+ - \frac{1}{2\pi}\mathscr{P}^+\,. \tag{28}$$

On imposing this gauge choice the constraints (26) can be "solved" by writing all quantities in terms of $(\mathscr{Q}^-(\tau), \mathscr{P}_-(\tau))$ and $(q_a(\tau, \sigma), p_a(\tau, \sigma))$, $a = 1, ..., d-2$, through the relations

$$p_+ = \frac{\pi}{\mathscr{P}_-}\left[p_a^2 + \frac{1}{(2\pi)^2}\left(\frac{\partial q_a}{\partial \sigma}\right)^2\right]\,,$$

$$\frac{\partial q^-}{\partial \sigma} = 2\pi \frac{p_a}{\mathscr{P}_-}\frac{\partial q_a}{\partial \sigma}\,. \tag{29}$$

Applying result (B) we find that the physical variables

$$q_a\,,\quad p_a\,,\quad \mathscr{Q}^{-*} = \mathscr{Q}^- - \mathscr{P}^-\tau\,,\quad \mathscr{P}_-^* = \mathscr{P}_- \tag{30a}$$

are described by the hamiltonian

$$H^* = \mathscr{P}_+\mathscr{P}_- = \pi\int_0^{2\pi} d\sigma\left[p_a^2 + \frac{1}{(2\pi)^2}\left(\frac{\partial q_a}{\partial \sigma}\right)^2\right]\,, \tag{30b}$$

agreeing with the result given in refs. [3,4].

The motivation for this work was to understand time-dependent gauge fixing more systematically and in illustrating our results we have confined ourselves to the well-known examples of the temporal and light-cone gauges for relativistic particles and strings respectively. The temporal gauge for strings [6] could be discussed similarly. Our results may also be of use in deriving hamiltonian formulations of Yang–Mills theories with time-dependent gauge choices. Exotic gauges such as the Poincaré or coordinate gauge $x^\mu A_\mu = 0$ have been discussed by a number of authors, particularly in relation to non-perturbative aspects of these theories – for references see ref. [7]. In addition to specific applications, there are a number of other directions in which this work might be continued. One obvious question is whether the problem of finding a suitable hamiltonian can be solved for gauge conditions and/or sets of physical variables of greater generality than those already discussed. It would also be interesting to express the results of this paper in more geometrical terms. Indeed, since symplectic geometry provides the natural mathematical language for classical mechanics (see ref. [8] for a readable introduction) this might prove to be the clearest way of attacking the problem in its most general form.

[#3] As before, $\mu = 0, ..., d-1$ and the metric convention is "mostly plus". Light-cone indices are defined by $v^\pm = (1/\sqrt{2})(v^0 \pm v^{d-1})$ and $v_\pm = (1/\sqrt{2})(v_0 \pm v_{d-1})$ for any vector.

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